Complex Numbers

Solve the equation $x^2+1=0$

Soln.
$$x^2+1=0$$
$$x^2=-1$$

A square of a real number is always positive. So where can we find a solution for this?

History: Mathematicians in the 16^{th} and 17^{th} century were trying to solve polynomial equations, namely,
$$f(x)=0$$
where f is a polynomial.

They were successful in finding formulas for roots/zeros of the equations when the degree of the polynomial is $n = 1, 2, 3, 4$.

Of course $n=1$ is trivial and for $n=2$ we know the celebrated quadratic formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Remarkably, Abel proved that it is impossible to find an algebraic formula for roots of equations with degree 5 or higher

During this endeavour mathematicians encountered equations like $x^2+1=0$ which has no real solutions. They had to accept the existence of $\sqrt{-1}$. Thus, mathematicians discovered/invented (don't know which one) complex numbers.

Imaginary Numbers

We denote $\sqrt{-1}$ by i. (i stands for imaginary).
we denote it by i to avoid the confusing erroneous identity given below:

$$-1 = (\sqrt{-1})^2 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{1} = 1$$

This step is not valid.

Henceforth, we use i to denote $\sqrt{-1}$ and we have the identity:

$$i^2 = -1.$$

Thus, we enforce the existence of a square root of -1.
so we get

$$\sqrt{-1} = \pm i$$

$$\left[(-i)^2 = -i \cdot -i = i^2 = -1 \right]$$

Def. The imaginary numbers are all real multiples of i, i.e., numbers of the form $a i$ where a is a real number.

Ex.

i

$2i$

$3i$

$\frac{1}{4} i$

$0.221 i$

πi.

Exercise.

Evaluate the following:

(i) $(2i)^2$

(ii) $5i + 7i$

(iii) $\left(\frac{1}{4}i\right)^3$

Soln. (i) $(2i)^2 = 2i \cdot 2i$

$= 4i^2$

$= 4(-1)$

$= -4$

(ii) $5i + 7i = 12i$

(iii) $\left(\frac{1}{4}i\right)^3 = \frac{1}{4}i \cdot \frac{1}{4}i \cdot \frac{1}{4}i$

$= \frac{1}{64} i \cdot i \cdot i$

$= \frac{1}{64} i^2 \cdot i$

$= \frac{1}{64} (-1) \cdot i$

$= -\frac{i}{64}$

Complex numbers

Def. Complex numbers are numbers of the form $a+bi$ where a and b are real numbers.

Examples. $2+3i$

$0+i = i$

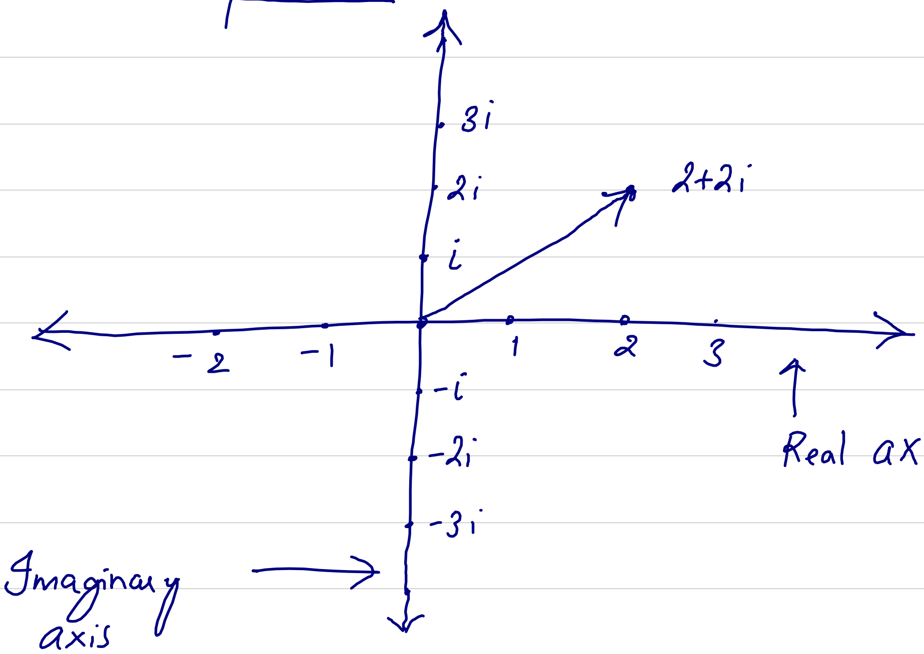
$0+7i = 7i$

$11+\sqrt{2}\,i$

$-3-\frac{1}{2}i$

Note that if we set $b=0$, then we get $a+0 \cdot i = a$. But a is a real number. Hence, all real numbers are complex numbers.

Geometric interpretation



Complex numbers can be thought of as points in the two dimensional plane.

Addition of complex numbers is simple:

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

Ex:

$$2+3i + 7+5i$$
$$= 2+7 + 3i+5i$$
$$= 9+8i$$

Due to syllabus restrictions we will not cover even the basic operations concerning complex numbers. Maybe you will learn them in higher courses.

Nevertheless, one cannot exaggerate the importance of complex numbers. Engineers definitely need complex numbers as fluids and aerodynamics models rely heavily on analysis of complex variables. But a natural question is

Ques Why would a biologist or an artist care about complex numbers?

Ans. Mathematics has been applied to biology for a long time and inorder to solve the open problems in the field concerning cell division, consciousness, language, origin of life, evolution etc, mathematics can play a major role like it has been doing in physics.

An artist should learn it because its beautiful.

Now we can solve quadratic equations which do not have real solutions:

Example Solve $2x^2 - 12x + 19 = 0$

Soln By quadratic formula,

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 2 \cdot 19}}{2 \cdot 2}$$

$$= \frac{12 \pm \sqrt{144 - 152}}{4}$$

$$= \frac{12 \pm \sqrt{-8}}{4}$$

$$= \frac{12 \pm \sqrt{8}\sqrt{-1}}{4}$$

$$= \frac{12 \pm \sqrt{8}\, i}{4}$$

$\therefore$ $x = \frac{12 + \sqrt{8}\, i}{4}$ and $x = \frac{12 - \sqrt{8}\, i}{4}$ are the solutions.

Complex Zeros

In this chapter our motivating for studying complex numbers is to find the zeros of polynomials That we were missing before.

Remember that we had a factorization like this:

$$P(x) = (x-1)(x+2)^2(x^2+1).$$

This polynomial has degree 5.

But the only zeros we could find were 1 (multipl. 1), and -2 (mult. 2). Since $P(x)$ has degree 5, counting multiplicities we are missing

$$5 - (1 + 2)$$

$$= 2 \text{ zeros}.$$

Since we have discovered / invented complex numbers we can obtain the two missing zeros:

$$x^2 + 1 = 0$$

$$\Rightarrow x^2 = -1$$

$$\Rightarrow x = \pm\sqrt{-1}$$

$$\Rightarrow x = \pm i$$

Thus $P(x)$ has 5 zeros counting multiplicities

1	Mult $-$ 1
-2	Mult $-$ 2
i	Mult. $-$ 1
$-i$	Mult $-$ 1
	5

Ques. How to factor x^2+1?

Ans. Since $x = i$ is a zero of x^2+1, $(x-i)$ is a factor. Since $x=-i$ is a zero of x^2+1, $(x+i)$ is a factor. Thus,

$$x^2+1 = (x-i)(x+i) \qquad \square$$

Therefore we have a complete factorization:

$$P(x) = (x-1)\,(x+2)^2\,(x-i)\,(x+i).$$

The above property is true in general and we have the following beautiful theorem:

<u>Theorem</u> (Gauss) Fundamental Theorem of Algebra.

Let $P(x)$ be a polynomial of degree n. $P(x)$ has exactly n zeros counted with multiplicities.

<u>Corollary</u>: Every polynomial $P(x)$ can be factored as a product of linear factors.

<u>Note</u>: Linear factors are of the form $x-a$ where a is a complex number.

Complex conjugation

Def. Let $a+bi$ be a complex number. The complex conjugate of $a+bi$ is the complex number $a-bi$.

Example.

The complex conjugate of $2+3i$ is $2-3i$

" " " " $-1+i$ is $-1-i$

" " " " $7-6i$ is $7+6i$

" " " " $-5-5i$ is $-5+5i$.

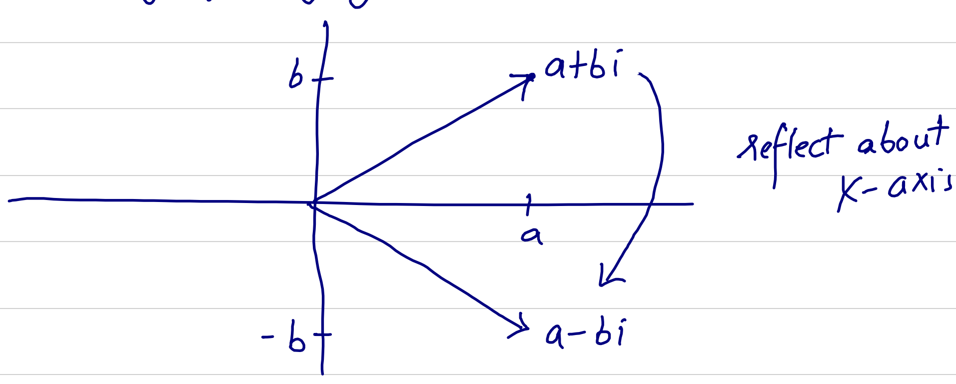
Note that if $b=0$, we get $a+bi = a$ where a is a real number. In this case the complex conjugate is just a itself because $b=0$.

We have the following theorem concerning conjugates of zeros:

Theorem.

Let $P(x)$ be a polynomial. If $a+bi$ is a zero of $P(x)$, then its complex conjugate $a-bi$ is also a zero of $P(x)$.

Geometric meaning of conjugate



Example:

Factor the polynomial $P(x) = x^4 - x^3 - 5x^2 - x - 6$ given that i is a zero of $P(x)$.

Solution. By the fundamental theorem of algebra there must be 4 zeros (counted with multiplicities).

Since i is a zero, by the complex conjugate theorem, the complex conjugate of i is also a zero. The complex conjugate of i is $-i$.

Hence, we have 2 zeros:

$$i \quad \text{and} \quad -i.$$

We need to find the other two.

Since i and $-i$ are zeros, $(x-i)$ and $(x+i)$ are factors. Recall that we can factor $P(x)$ completely. Let

$$P(x) = (x-i)(x+i)(x-c)(x-d)$$

where c and d are complex numbers we need to find. (c could be equal to d).

Multiply $(x-i)(x+i)$

$$= x(x+i) - i(x+i)$$

$$= x^2 + \cancel{ix} - \cancel{ix} - i^2$$

$$= x^2 + 1 \qquad (i^2 = -1)$$

Thus,

$$P(x) = (x^2+1)(x-c)(x-d).$$

Let's divide $P(x)$ by x^2-1, Then we have

$$\frac{P(x)}{x^2+1} = (x-c)(x-d).$$

$$
\begin{array}{r|l}
x^2+1 & x^4 - x^3 - 5x^2 - x - 6 \quad (x^2 - x - 6 \\
 & \underline{\overset{(-)}{x^4} \qquad \overset{(-)}{+x^2}} \\
 & -x^3 - 6x^2 - x - 6 \\
 & \underline{\overset{(+)}{-x^3} \qquad \overset{(+)}{-x}} \\
 & -6x^2 - 6 \\
 & \underline{\overset{(+)}{-6x^2} \qquad \overset{(+)}{-6}} \\
 & 0
\end{array}
$$

Thus, $P(x) = \underbrace{(x^2+1)}_{(x+i)(x-i)} \, (x^2 - x - 6)$. $(*)$

Lets factor $x^2 - x - 6$ $\qquad (-3)(2) = -6$

$= (x-3)(x-1)$ $\qquad -3 + 2 = -1$

Hence, by $(*)$,

$$P(x) = (x+i)(x-i)(x-3)(x-1).$$

Exercise:

Factor $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$ given that $x - 2i$ is a factor.

Soln. Since $x-2i$ is a factor, $2i$ is a zero.

The complex conjugate of $2i$ is $-2i$.

Thus, $2i$ and $-2i$ are zeros.

By the fundamental thm of algebra, $P(x)$ has exactly 4 zeros (with multiplicity).

Hence, $P(x) = \underbrace{(x-2i)}_{(2i \text{ is zero})}\underbrace{(x+2i)}_{(-2i \text{ is zero})}(x-c)(x-d)$

where c and d are complex numbers that we need to find.

Now $(x-2i)(x+2i) = x^2 + 2ix - 2ix - 4i^2$

$= x^2 + 4$

Thus, $\dfrac{P(x)}{x^2+4} = (x-c)(x-d)$

$$
\begin{array}{r|l}
x^2+4 & x^4 - 3x^3 + 6x^2 - 12x + 8 \;\big(\; x^2 - 3x + 2 \\
 & \underset{(-)}{x^4} \quad\quad \underset{(-)}{+4x^2} \\
\hline
 & -3x^3 + 2x^2 - 12x + 8 \\
 & \underset{(+)}{-3x^3} \quad\quad \underset{(+)}{-12x} \\
\hline
 & 2x^2 + 8 \\
 & \underset{(-)}{2x^2} \; \underset{(-)}{+8} \\
\hline
 & 0
\end{array}
$$

Thus, $P(x) = \underbrace{(x^2+4)}_{= (x-2i)(x+2i)}(x^2 - 3x + 2)$

Now factor $x^2 - 3x + 2$. $\quad -2, -1$

$= (x-2)(x-1)$

$\therefore P(x) = (x-2i)(x+2i)(x-2)(x-1)$ $\square$

Bonus hw **Exercise** (1) Factor the polynomial $P(x) = x^4 - 2x^3 + x^2 + 2x - 2$ given that $1+i$ is a zero.

Exercise (2) Factor the polynomial $P(x) = x^4 - 2x^2 + 16x - 15$ given that $1+2i$ is a zero.

Example

Factor the polynomial

$$P(x) = x^5 + 2x^4 - x - 2.$$

Note: The book ambiguously distinguishes real and complex zeros. All real numbers are complex numbers. But not all complex numbers are real numbers.

Solution. Let's use Descartes' rule:

$$P(x) = \underbrace{x^5 + 2x^4 - x}_{\text{1 sign change}} - 2$$

There is 1 positive zero

$$P(-x) = -x^5 + 2x^4 + x - 2$$

2 sign changes

There are either 2 or 0 negative zeros.

Possibilities:

	Positive real zeros	Negative real zeros	Non real complex zeros.	Sum
(1)	1	0	3	= 4
(2)	1	2	2	= 4

Use the rational zero theorem:

$a_0 = -2, \quad a_n = 1$

Rational zero $= \dfrac{\text{factor of } a_0}{\text{factor of } a_n}$

$= \dfrac{\text{factor of } -2}{\text{factor of } 1}$

$= \dfrac{\{\pm 1, \pm 2\}}{\{\pm 1\}}$

possible rational zeros: $1, -1, 2, -2$.

Test the zeros:

1?

$$\begin{array}{r|rrrrrr} 1 & 1 & 2 & 0 & 0 & -1 & -2 \\ & & 1 & 3 & 3 & 3 & 2 \\ \hline & 1 & 3 & 3 & 3 & 2 & \boxed{0} \end{array}$$

1 is a zero $\Rightarrow$ $(x-1)$ is a factor

No need to test for 2.

-1?

$$\begin{array}{r|rrrrr} -1 & 1 & 3 & 3 & 3 & 2 \\ & & -1 & -2 & -1 & -2 \\ \hline & 1 & 2 & 1 & 2 & \boxed{0} \end{array}$$

-1 is a zero $\Rightarrow$ $(x+1)$ is a factor.

Because there is atleast one $-$ve zero, -2 must be a zero by Descartes' rule.

Thus, -2 is a zero $\Rightarrow$ $(x+2)$ is a factor.

$$
\begin{array}{r|rrrr}
-2 & 1 & 2 & 1 & 2 \\
 & & -2 & 0 & -2 \\
\hline
 & 1 & 0 & 1 & \boxed{0}
\end{array}
$$

Thus $P(x) = (x-1)(x+1)(x+2)(x^2+1)$

But we Know from previous work:

$$x^2+1 = (x+i)(x-i)$$

$\therefore$ $P(x) = (x-1)(x+1)(x+2)(x+i)(x-i)$.